On the birational unboundedness of higher dimensional Q-Fano varieties

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ON THE BIRATIONAL UNBOUNDEDNESS OF HIGHER DIMENSIONAL $\mathbb{Q}$-FANO VARIETIES

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ABSTRACT. We show that the family of ($\mathbb{Q}$-factorial and log terminal) $\mathbb{Q}$-Fano n-folds with Picard number one is birationally unbounded for $n \geq 6$.

1. INTRODUCTION

In this paper, we say that a normal projective variety defined over the field $\mathbb{C}$ of complex numbers is a $\mathbb{Q}$-*Fano variety* if it is $\mathbb{Q}$-factorial, has only log terminal singularities and its anticanonical divisor is ample. $\mathbb{Q}$-Fano varieties appear as one of the final outcomes of the log Minimal Model Program and play an important role in the classification of algebraic varieties. Because of its importance, boundedness of ($\mathbb{Q}$-)Fano varieties has been studied by many authors. For example, Kollár-Miyaoka-Mori [7] proved the boundedness of smooth Fano varieties in arbitrary dimension. Kollár-Miyaoka-Mori-Takagi [8] proved the boundedness of $\mathbb{Q}$-Fano threefolds with canonical singularities. McKernan [11] proved the boundedness of log terminal Fano pairs of bounded index.

In dimension two, $\mathbb{Q}$-Fano varieties, which are usually called log Del Pezzo surfaces, are unbounded but they are birationally bounded because they are all rational. Here we give the definition of birational boundedness.

Definition 1.1. A class of varieties $\mathfrak{V}$ is *birationally bounded* if there exists a morphism $f\colon \mathcal{X} \to \mathcal{S}$ between algebraic schemes such that every variety in $\mathfrak{V}$ is birational to one of the geometric fibers of f. We say that $\mathfrak{V}$ is *birationally unbounded* if it is not birationally bounded.

Lin [10] proved the birational unboundedness of $\mathbb{Q}$-Fano threefolds with Picard number one. It seems difficult to generalize the proof given by Lin to higher dimensional cases because it depends on the Sarkisov Program which is build upon the Minimal Model Program. Following is the main theorem of this paper.

Theorem 1.2. *If $n \geq 6$ then the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally unbounded.*

In order to get the boundedness results of $\mathbb{Q}$-Fano varieties, we need to impose a restriction on some invariants. A. Borisov, L. Borisov and Alexeev independently proposed the following interesting conjecture (cf. [2], [1]).

Conjecture 1.3 (Borisov-Alexeev-Borisov)**.** *Fix $\varepsilon > 0$. Then the family of all $\mathbb{Q}$-Fano varieties of a given dimension with log canonical discrepancy greater than ε is bounded.*

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This conjecture is solved only in special cases. Alexeev [1] and Nikulin [12] proved Conjecture 1.3 in dimension two, and A. Borisov-L. Borisov [3] proved Conjecture 1.3 in the toric case. Theorem 1.2, as well as Lin's result [10], shows that we cannot drop the restriction on ε in the hypothesis of Conjecture 1.3 even if we replace the boundedness by the birational boundedness.

In [13], we constructed examples of non-ruled $\mathbb{Q}$-Fano weighted hypersurfaces with Picard number one. The proof of Theorem 1.2 will be done by showing that these examples are birationally unbounded if the dimension is greater than or equal to 6.

This paper is organized as follows. In Section 2, we recall examples of non-ruled $\mathbb{Q}$-Fano weighted hypersurfaces $X = X_f$ from [13] and study their properties. In particular, we recall that, if X is defined over an algebraically closed field $\Bbbk$ of characteristic two, there is a big line bundle $\mathcal{L}$ on Y which is a subsheaf of Ω_Y^{n-1}, where Y is a nonsingular model of X and n is the dimension of Y. We prove the birational invariance of the global sections of $\mathcal{L}$ under a suitable condition. In Section 3, we construct a "large" birationally trivial family of $\mathbb{Q}$-Fano weighted hypersurfaces defined over $\Bbbk$ assuming that the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally bounded. Finally, in Section 4, we compute the dimension of the birationally trivial family of weighted hypersurfaces defined over $\Bbbk$ and show that it is not so "large" compared with the one obtained in Section 3, which completes the proof of our main theorem.

Notations and conventions.

- Let V be a vector space. We say that an element $v \in V$ is *general* (resp. *very general*) if it belongs to the complement of a suitable proper closed subspace (resp. at most countable union of suitable proper closed subspaces) of V.
- For a vector space V, we denote by $\mathbb{P}(V)$ (resp. $\mathbb{P}_{\mathrm{sub}}(V)$) the projective space parametrizing one dimensional quotients (resp. subspaces) of V.
- For a $\mathbb{Q}$-divisor D on a variety X, we denote by $\llcorner D \lrcorner$ the round down of D.
- Let $\varphi \colon Y \to X$ be a morphism between normal varieties and D a $\mathbb{Q}$-Cartier Weil divisor on X. We denote by $\varphi^* D$ the pull back of D as a $\mathbb{Q}$-divisor, that is, $\varphi^* D := (1/m)\varphi^*(mD)$, where m is a positive integer such that mD is a Cartier divisor on X.
- Let $\mathcal{L}$ (resp. D) be a reflexive sheaf of rank one (resp. Weil divisor) on a normal variety X with the finite dimensional global sections. We denote by $|\mathcal{L}|$ (resp. $|D|$) the complete linear system $\mathbb{P}_{\mathrm{sub}}(H^0(X, \mathcal{L}))$ (resp. $\mathbb{P}_{\mathrm{sub}}(H^0(X, \mathcal{O}_X(D)))$. We denote by
$$\Phi_{|\mathcal{L}|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L})) \text{ (resp. } \Phi_{|D|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D))))$$
the rational map which is associated with the complete linear system $|\mathcal{L}|$ (resp. $|D|$).
- Let S be a graded ring and $f \in S$ a homogeneous element. By $(f = 0) \subset \operatorname{Proj} S$, we mean the closed subscheme defined by the homogeneous ideal generated by f.

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2. Non-ruled weighted hypersurfaces

We recall examples of non-ruled $\mathbb{Q}$-Fano weighted hypersurfaces. Let a, l, m and n be positive integers, where a and l are odd. Put $b = (al-1)/2$.

Definition 2.1. Let X be a variety defined over a field k. We say that X is *ruled* (resp. *separably uniruled*) if there exist a variety Y defined over k of dimension $\dim X - 1$ and a birational map (resp. separable dominant map) $Y \times \mathbb{P}^1 \dashrightarrow X$.

Definition 2.2. Let l, m, n, a and b be as above and let k be a field. We denote simply by $k[x_0, \dots, x_n]$ and $k[x_0, \dots, x_n, y]$ the graded rings whose gradings are given by $\deg x_i = 1$ for $i = 0, \dots, m$, $\deg x_i = a$ for $i = m+1, \dots, n$ and $\deg y = b$. We define weighted projective spaces P_k and Q_k as follows.

- $P_k = \mathbb{P}_k(\overbrace{1, \dots, 1}^{m+1}, \overbrace{a, \dots, a}^{n-m}, b) := \operatorname{Proj} k[x_0, \dots, x_n, y]$.
- $Q_k = \mathbb{P}_k(\overbrace{1, \dots, 1}^{m+1}, \overbrace{a, \dots, a}^{n-m}) := \operatorname{Proj} k[x_0, \dots, x_n]$.

For a positive integer d, we denote by $H_d(k)$ the k-vector space $k[x_0, \dots, x_n]_d$, the degree d part of the graded ring $k[x_0, \dots, x_n]$. For an element $f \in H_{al}(k)$, we define

$$X_f := (y^2 x_0 - f(x_0, \dots, x_n) = 0) \subset P_k.$$

We consider the following condition on l, m and n.

Condition 2.3.

(1) m, n are integers and l is an odd integer.
(2) $4 \le n$ and $0 < m < n$.
(3) $n - m + 1 < l < 2(n - m)$.

Theorem 2.4 ([13], Theorem 7.3 and 1.3)**.** *Let l, m and n be integers which satisfy* Condition 2.3. *Then, the following assertions hold for every odd positive integer a with $a > (m+1)/2$.*

(1) *The weighted hypersurface $X_f \subset P_{\mathbb{C}}$ of degree al defined over $\mathbb{C}$ is a non-ruled $\mathbb{Q}$-Fano variety with Picard number one for a very general $f \in H_{al}(\mathbb{C})$.*
(2) *The weighted hypersurface $X_f \subset P_{\Bbbk}$ of degree al defined over an algebraically closed field $\Bbbk$ of characteristic two is not separably uniruled for a general $f \in H_{al}(\Bbbk)$.*

Remark 2.5. In [13, Theorem 7.3], we did not mention the Picard number of X_f. In our case, a general weighted hypersurface X_f defined over $\mathbb{C}$ is quasi smooth (cf. [13, Lemma 3.5]). Hence it follows from [4, Theorem 3.2.4] that the Picard number of X_f is one.

Throughout the paper,

- we fix positive integers l, m and n which satisfy Condition 2.3,
- a is an odd integer with $a > m+1$, and
- $b = (al-1)/2$.

Throughout the present section, we work over an algebraically closed field $\Bbbk$ of characteristic two and we fix a general element $f = f(x_0, \dots, x_n) \in H_{al}(\Bbbk)$. Put $X = X_f, P = P_{\Bbbk}$ and $Q = Q_{\Bbbk}$. Let $\pi_P \colon P \dashrightarrow Q$ be the natural projection and $\pi \colon X \dashrightarrow Q$ be its restriction. The rational maps π_P and π are defined outside the point $\mathsf{p} := (0 : \cdots : 0 : 1) \in P$.

Lemma 2.6 ([13], Lemma 3.10)**.** *The following assertions hold.*

(1) *$X \cap \mathrm{D}_+(x_0)$ has only isolated singularities which are isomorphic to the singularities of the origin of the hypersurface defined by the equation*
$$\nu^2 = \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-1}\xi_n + h(\xi_1, \dots, \xi_n)$$
if n is even, or
$$\nu^2 = \alpha\xi_1^2 + \xi_2\xi_3 + \xi_4\xi_5 + \cdots + \xi_{n-1}\xi_n + \xi_1^3 + h'(\xi_1, \dots, \xi_n)$$
if n is odd, where $\alpha \in \Bbbk$, $\deg h, \deg h' \geq 3$ and the coefficient of ξ_1^3 in h' is zero.

(2) *Put $X_{\mathrm{qs}} := X \setminus (\mathrm{Sing}(X) \cap \mathrm{D}_+(x_0))$ and $U_{\mathrm{qs}} := X_{\mathrm{qs}} \cap \mathrm{D}_+(x_0 \cdots x_n y)$. Then, $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$ is a toroidal embedding without self-intersection.*

Lemma 2.7 ([13], Corollary 3.13)**.** *There is a resolution $r \colon Y \to X$ of singularities of X with the following properties.*

(1) *Around the isolated singular point which is contained in $X \cap \mathrm{D}_+(x_0)$, r is a blow up at the point.*

(2) *$r|_{r^{-1}(X_{\mathrm{qs}})} \colon r^{-1}(X_{\mathrm{qs}}) \to X_{\mathrm{qs}}$ is a resolution of the toroidal embedding $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$.*

We fix a resolution $r \colon Y \to X$ of singularities of X which is obtained by Lemma 2.7. Let V be the smooth locus of Q, $U := (\pi_P|_{P \setminus \{\mathsf{p}\}})^{-1}(V)$ and $X^\circ := X \cap U$. We see that U is smooth and $\operatorname{codim}_X(X \setminus X^\circ) \geq 2$. We denote by $\pi^\circ \colon X^\circ \to V$ the restriction of $\pi \colon X \dashrightarrow Q$.

Lemma 2.8 ([13], Lemma 4.2)**.** *Notation as above.*

(1) *There is an exact sequence*
$$0 \to (\pi^\circ)^*\Omega_V^1 \to \Omega_U^1|_{X^\circ} \to \mathcal{O}_{X^\circ}(-b) \to 0.$$

(2) *There is an exact sequence*
$$0 \to \mathcal{O}_{X^\circ}(-al) \xrightarrow{\delta} \Omega_U^1|_{X^\circ} \to \Omega_{X^\circ}^1 \to 0,$$
and we have $\operatorname{Im}\delta \subset (\pi^\circ)^*\Omega_V^1$.

(3) *There is an exact sequence*
$$0 \to \operatorname{Coker}[\mathcal{O}_{X^\circ}(-al) \xrightarrow{\delta} (\pi^\circ)^*\Omega_V^1] \to \Omega_{X^\circ}^1 \to \mathcal{O}_{X^\circ}(-b) \to 0.$$

Definition 2.9. Let $\mathcal{M}^\circ$ be the double dual of

$$\bigwedge^{n-1}\left(\mathrm{Coker}[\mathcal{O}_{X^\circ}(-al) \xrightarrow{\delta} (\pi^\circ)^*\Omega^1_V]\right)$$

and $\mathcal{M} = i_*\mathcal{M}^\circ$, where $i\colon X^\circ \hookrightarrow X$ is the embedding.

It follows from Lemma 2.8 that $\mathcal{M} \subset (\Omega_X^{n-1})^{\vee\vee}$ and $\mathcal{M} \cong \mathcal{O}_X(A)$, where

$$A := (l+m-n)a - (m+1).$$

By Condition 2.3 and the assumption $a > m+1$, we have $a < A < b$.

Definition 2.10. We denote by $\{E_j \mid j \in J\}$ the set of exceptional divisors of $r\colon Y \to X$ which are obtained by resolving the singularities of the toroidal embedding $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$, and put $E = \cup_{j\in J} E_j$.

Let M be a Weil divisor on X such that $\mathcal{O}_X(M) \cong \mathcal{M}$. Then, by [13, Lemma 4.5], there is an injection $\mathcal{O}_Y(\llcorner r^*M \lrcorner)|_{Y\setminus E} \hookrightarrow \Omega_Y^{n-1}|_{Y\setminus E}$.

Definition 2.11. For each $j \in J$, we denote by $c_j(M)$ the largest integer c_j such that the injection

$$\mathcal{O}_Y(\llcorner r^*M \lrcorner)|_{Y\setminus E} \hookrightarrow \Omega_Y^{n-1}|_{Y\setminus E}$$

lifts to an injection

$$\mathcal{O}_Y(\llcorner r^*M \lrcorner + c_jE_j)|_{U_j} \hookrightarrow \Omega_Y^{n-1}|_{U_j},$$

where $U_j = Y \setminus \bigcup_{k\in J\setminus\{j\}} E_k$. We denote by $\mathcal{L}$ the image of the injection

$$\mathcal{O}_Y\left(\llcorner r^*M \lrcorner + \sum\nolimits_{j\in J} c_j(M)E_j\right) \hookrightarrow \Omega_Y^{n-1}.$$

The definition of $\mathcal{L}$ does not depend on the choice of M. We see that $\mathcal{L}$ is an invertible sheaf which is a subsheaf of Ω_Y^{n-1}. Notice that the sheaf $\mathcal{L}$ coincides with the one defined in [13, Definition 4.4].

Lemma 2.12. *For each $j \in J$, the integer $c_j(M)$ is nonnegative.*

Proof. Let α be a sufficiently divisible positive integer such that αM is a Cartier divisor on X. Then, for suitable rational numbers c'_j, we can write

$$\mathcal{L}^{\otimes\alpha} \cong r^*\mathcal{O}_X(\alpha M) \otimes \mathcal{O}_Y\left(\sum\nolimits_{j\in J} \alpha c'_j E_j\right).$$

We see that $c_j(M) = \lceil c'_j \rceil$. Hence it is enough to show that $c'_j > -1$. Let $x \in X$ be a point which is contained in the center of E_j. By [13, Lemma 5.7] or [13, Lemma 5.8], there are a function $h \in \mathcal{O}_{X,x}$ vanishing along the center of E_j and a rational $(n-1)$-form ω such that $\mathcal{O}_X(\alpha M)_x \subset \mathcal{O}_{X,x} \cdot h\omega^{\otimes\alpha}$. Moreover, the rational $(n-1)$-form ω is of the form

$$\omega = \frac{dg_1 \wedge \cdots \wedge dg_{n-1}}{g_1 \cdots g_{n-1}},$$

where $g_i \in \mathcal{O}_{X,x}$, so that the order of the pole of $r^*\omega$ along E_j is at most one. By the definition of c'_j, the integer $\alpha c'_j$ is not less than the order of the zero of the rational $(n-1)$-form $r^*(h\omega^{\otimes\alpha})$. Thus, we see that

$$\alpha c'_j \geq -\alpha + \mathrm{mult}_{E_j}(r^*h) > -\alpha.$$

This completes the proof. $\square$

We see that every global section of $\mathcal{M}$ lifts uniquely to a global section of $\mathcal{O}_Y(\llcorner r^*M \lrcorner)$. By Lemma 2.12, $H^0(Y, \mathcal{O}_Y(\llcorner r^*M \lrcorner))$ is naturally isomorphic to $H^0(Y, \mathcal{L})$. Therefore, we can identify the rational map $\Phi_{|\mathcal{L}|} \colon Y \dashrightarrow \mathbb{P}(H^0(Y, \mathcal{L}))$ with the composite of $r \colon Y \to X$ and $\Phi_{|\mathcal{M}|} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{M}))$. The rational map $\Phi_{|\mathcal{M}|}$ is the composite of the projection $\pi \colon X \dashrightarrow Q$ and $\Phi_{|\mathcal{O}_Q(A)|}$ since $\mathcal{M} \cong \mathcal{O}_X(A)$ and $a < A < b$. Let Z be the image of the rational map $\Phi_{|\mathcal{L}|}$. It follows from the argument above that Z coincides with the images of $\Phi_{|\mathcal{M}|}$ and $\Phi_{|\mathcal{O}_Q(A)|}$.

Lemma 2.13. *Notation as above. There is a commutative diagram*

$$\begin{tikzcd}
Y \arrow[r, "r"] \arrow[dr, dotted, "\Phi_{|\mathcal{L}|}"'] & X \arrow[r, dotted, "\pi"] \arrow[d, dotted, "\Phi_{|\mathcal{M}|}"] & Q \arrow[dl, dotted, "\Phi_{|\mathcal{O}_Q(A)|}"] \\
& Z &
\end{tikzcd}$$

and the rational map $\Phi_{|\mathcal{O}_Q(A)|} \colon Q \dashrightarrow Z$ is a birational map. Moreover, its inverse $\Phi^{-1}_{|\mathcal{O}_Q(A)|} \colon Z \dashrightarrow Q$ is the blow up of Q along the subvariety $(x_0 = \cdots = x_m = 0)$.

Proof. The existence of the commutative diagram follows from the preceding argument. The map $\Phi_{|\mathcal{O}_Q(A)|} \colon Q \dashrightarrow Z$ is birational since we have $a < A$. Let $\widetilde{Q} \to Q$ be the blow up of Q along the subvariety $(x_0 = \cdots = x_m = 0)$. It is straightforward to check that the induced rational map $\widetilde{Q} \dashrightarrow Z$ is everywhere defined and it is biregular at each point of the exceptional divisor. This shows that Z is isomorphic to $\widetilde{Q}$. $\square$

Next, we shall prove the birational invariance of the global sections of $\mathcal{L}$ under an additional condition on l, m and n.

Lemma 2.14. *If l, m and n satisfy $l + 2m - 2n + 2 \leq 0$ in addition to* Condition 2.3 *then we have $H^0(X, \mathcal{M}) = H^0\left(X, (\Omega_X^{n-1})^{\vee\vee}\right)$.*

Proof. Let U be an open subset of X. By using local cohomology, we see that $H^i(X, \mathcal{O}_X(j)) \cong H^i(U, \mathcal{O}_X(j))$ for $i \leq \operatorname{codim}_X(X \setminus U) - 2$ and any j.

Let V be a smooth open subset of Q such that the open subset $U := \pi^{-1}(V)$ of X is smooth and $\operatorname{codim}_Q(Q \setminus V) \geq 2$. Let $\mathcal{F}$ be the cokernel of the map $\delta \colon \mathcal{O}_U(-al) \to \pi^*\Omega_V^1$, which sits in the exact sequence

$$0 \to \mathcal{F} \to \Omega_U^1 \to \mathcal{O}_U(-b) \to 0.$$

After shrinking V and U, we may assume that $\mathcal{F}$ is locally free on U and we still have $\operatorname{codim}_Q(Q \setminus V) \geq 2$ because a torsion free sheaf on a smooth variety is locally free outside a closed subset of codimension ≥ 2. Taking the wedge product $\bigwedge^{n-1}$, we obtain the exact sequence

$$0 \to \mathcal{M}|_U \to \Omega_U^{n-1} \to \mathcal{O}_U(-b) \otimes \bigwedge^{n-2} \mathcal{F} \to 0.$$

We have an isomorphism

$$\mathcal{O}_U(-b) \otimes \bigwedge^{n-2} \mathcal{F} \cong \mathcal{O}_U(-b) \otimes \bigwedge^{n-1} \mathcal{F} \otimes \mathcal{F}^\vee = \mathcal{O}_U(-b) \otimes \mathcal{M}|_U \otimes \mathcal{F}^\vee.$$

Put

$$\mathcal{G} := i_*(\mathcal{O}_U(-b) \otimes \mathcal{M}|_U \otimes \mathcal{F}^\vee) \cong i_*(\mathcal{O}_U(A') \otimes \mathcal{F}^\vee),$$

where $i\colon U \hookrightarrow X$ is the open immersion and $A' = A - b$. Then the assertion $H^0(X, \mathcal{M}) \cong H^0(X, (\Omega_X^{n-1})^{\vee\vee})$ follows from the assertion $H^0(X, \mathcal{G}) = 0$.

Now let U be an open subset of X such that it is contained in the smooth locus of X, the sheaf $\mathcal{F}$ is locally free on U and we have $\operatorname{codim}_X(X \setminus U) \geq 3$. We can take such an open set U because a reflexive sheaf on a smooth variety is locally free outside a closed subset of codimension ≥ 3. Let U' be an open subset of P such that it is contained in the smooth locus of P and $U' \cap X = U$. By the exact sequence

$$0 \to \mathcal{O}_U(b) \to T_U \to \mathcal{F}^\vee \to 0,$$

we obtain an exact sequence

$$0 \to \mathcal{O}_U(A) \to \mathcal{H}|_U \to \mathcal{G}|_U \to 0,$$

where $\mathcal{H} := i_*(T|_U \otimes \mathcal{O}_U(A'))$. As $H^1(U, \mathcal{O}_U(A)) \cong H^1(X, \mathcal{O}_X(A)) = 0$, this shows that the assertion $H^0(X, \mathcal{G}) = 0$ is equivalent to the assertion

$$H^0(X, \mathcal{O}_X(A)) \cong H^0(X, \mathcal{H}).$$

By the exact sequence

$$0 \to T_U \to T_{U'}|_U \to \mathcal{O}_U(al) \to 0,$$

we obtain an exact sequence

$$0 \to \mathcal{H}|_U \to \mathcal{H}'|_U \to \mathcal{O}_U(A' + al) \to 0,$$

where $\mathcal{H}' := i_*((T_{U'}|_U) \otimes \mathcal{O}_U(A'))$. To conclude that $H^0(X, \mathcal{G}) = 0$, it is enough to show that $h^0(X, \mathcal{H}') = h^0(X, \mathcal{O}_X(A))$. By the exact sequence

$$0 \to \mathcal{O}_U \to \mathcal{O}_U^{\oplus m+1}(1) \bigoplus \mathcal{O}_U(a)^{\oplus n-m} \bigoplus \mathcal{O}_U(b) \to T_{U'}|_U \to 0,$$

we obtain an exact sequence

$$0 \to \mathcal{O}_U(A') \to \mathcal{O}_U(1 + A')^{\oplus m+1} \bigoplus \mathcal{O}_U(a + A')^{\oplus n-m} \bigoplus \mathcal{O}_U(A) \to \mathcal{H}'|_U \to 0.$$

By the assumption, we have

$$1 + A' < a + A' = ((l + 2m - 2n + 2)a - (2m + 1))/2 < 0.$$

Thus, we have $H^0(X, \mathcal{H}') \cong H^0(X, \mathcal{O}_X(A))$ since we have

$$H^1(U, \mathcal{O}_U(A)) \cong H^1(X, \mathcal{O}_X(A)) = 0.$$

Therefore, we have $H^0(X, \mathcal{H}) \cong H^0(X, \mathcal{O}_X(A))$. This completes the proof. □

Proposition 2.15. *If l, m and n satisfy $l + 2m - 2n + 2 \leq 0$ in addition to* Condition 2.3 *then we have $H^0(Y, \mathcal{L}) = H^0(Y, \Omega_Y^{n-1})$.*

Proof. Let F be the exceptional locus of $\varphi\colon Y \to X$ and let $\omega \in H^0(Y, \Omega_Y^{n-1})$. Then, we see that

$$\omega|_{Y \setminus F} \in H^0(X \setminus \varphi(F), (\Omega_X^{n-1})^{\vee\vee}) = H^0(X \setminus \varphi(F), \mathcal{M}) = H^0(X, \mathcal{M}).$$

Thus, we have $\omega|_{Y \setminus F} \in H^0(Y \setminus F, \mathcal{L})$. It follows from the definition of $\mathcal{L}$ that $\omega \in H^0(Y, \mathcal{L})$. □

3. Construction of birationally trivial families

We begin with defining the scheme which parametrizes birational correspondences between members of two families. Let $\phi_1 \colon \mathcal{X}_1 \to \mathcal{S}_1$ and $\phi_2 \colon \mathcal{X}_1 \to \mathcal{S}_2$ be projective morphisms between noetherian schemes defined over an algebraically closed field. Let $\mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2)$ be the relative Hilbert scheme of $\Phi := (\phi_1, \phi_2) \colon \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{S}_1 \times \mathcal{S}_2$ and $\pi \colon \mathcal{Z} \to \mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2)$ be the universal family of subschemes. We have the following diagram.

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook, "\iota"] \arrow[dr, "\pi"'] & (\mathcal{X}_1 \times \mathcal{X}_2) \times_{(\mathcal{S}_1 \times \mathcal{S}_2)} \mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2) \arrow[r, "p_{12}"] \arrow[d, "p_3"] & \mathcal{X}_1 \times \mathcal{X}_2 \arrow[d, "\Phi"] \\
& \mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2) \arrow[r, "p"] & \mathcal{S}_1 \times \mathcal{S}_2.
\end{tikzcd}$$

In the diagram above, p_{12}, p_3 are the natural projections and ι is the closed embedding. Let $q_i \colon \mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2) \to \mathcal{S}_i$ be the composite of p and the natural projection $\mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_i$.

Now let us assume that both ϕ_1 and ϕ_2 are flat morphisms between varieties with geometrically integral fibers. Then the set

$$\{\, t \in \mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2) \mid \mathcal{Z}_t \text{ is a birational correspondence} \,\}$$

is open in $\mathrm{Hilb}(\mathcal{X}_1 \times \mathcal{X}_2/\mathcal{S}_1 \times \mathcal{S}_2)$. Here we say that $\mathcal{Z}_t \subset (\mathcal{X}_1)_{q_1(t)} \times (\mathcal{X}_2)_{q_2(t)}$ is a *birational correspondence* if it is a geometrically integral subscheme of $(\mathcal{X}_1)_{q_1(t)} \times (\mathcal{X}_2)_{q_2(t)}$ such that the projection $(\mathcal{X}_1)_{q_1(t)} \times (\mathcal{X}_2)_{q_2(t)} \to (\mathcal{X}_i)_{q_i(t)}$ restricts to a birational morphism $\mathcal{Z}_t \to (\mathcal{X}_i)_{q_i(t)}$ for $i = 1, 2$. For the proof of the openness, we refer the reader to [5, Proposition 1.3.2] where the settings are slightly different from ours. Our case can be proved similarly. We denote by $\mathrm{Bir}(\mathcal{X}_1/\mathcal{S}_1, \mathcal{X}_2/\mathcal{S}_2)$ the set above with the open subscheme structure.

Definition 3.1. We call $\mathrm{Bir}(\mathcal{X}_1/\mathcal{S}_1, \mathcal{X}_2/\mathcal{S}_2)$ the *scheme parametrizing birational correspondences between members of $\mathcal{X}_1/\mathcal{S}_1$ and $\mathcal{X}_2/\mathcal{S}_2$*. We define

$$\Gamma(\mathcal{X}_1/\mathcal{S}_1, \mathcal{X}_2/\mathcal{S}_2) := \pi^{-1}(\mathrm{Bir}(\mathcal{X}_1/\mathcal{S}_1, \mathcal{X}_2/\mathcal{S}_2))$$

and call it *the universal family of birational correspondences between members of $\mathcal{X}_1/\mathcal{S}_1$ and $\mathcal{X}_2/\mathcal{S}_2$*.

Definition 3.2. Let l, m and n be integers which satisfy Condition 2.3 and we fix them. For an odd integer $a > m + 1$, let $\mathcal{X}_a \to \mathcal{S}_a$ be the family of quasi smooth weighted hypersurfaces X_f of degree al in $P_{\mathbb{C}}$ (cf. Definition 2.2), and let $\mathcal{X}'_a \to \mathcal{S}'_a$ be the family of weighted hypersurfaces X'_f of degree al in $P_{\Bbbk}$ with the singularities described in Lemma 2.6.

We see that $\mathcal{S}_a$ and $\mathcal{S}'_a$ are open subsets of

$$\mathbb{P}_{\mathrm{sub}}(H^0(Q_{\mathbb{C}}, \mathcal{O}_{Q_{\mathbb{C}}}(al))) \text{ and } \mathbb{P}_{\mathrm{sub}}(H^0(Q_{\Bbbk}, \mathcal{O}_{Q_{\Bbbk}}(al)))$$

respectively so that we have

$$\dim \mathcal{S}_a = h^0(Q_{\mathbb{C}}, \mathcal{O}_{Q_{\mathbb{C}}}(al)) - 1 = h^0(Q_{\Bbbk}, \mathcal{O}_{Q_{\Bbbk}}(al)) - 1 = \dim \mathcal{S}'_a.$$

In the following argument, we will freely replace $\mathcal{S}_a$ and $\mathcal{S}'_a$ by their open subsets.

Definition 3.3. We say that a family of varieties is *birationally trivial* if every two members of the family are birational.

Proposition 3.4. *Suppose that the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally bounded. Then, there exists a constant R such that, for every odd positive integer a with $a > m+1$ and a general point $s_a \in \mathcal{S}_a$, there is a closed subvariety $\mathcal{B}_a$ of $\mathcal{S}_a$ with the following properties.*

(1) *$\mathcal{B}_a$ parametrizes a birationally trivial family.*
(2) *$\mathcal{B}_a$ passes through s_a.*
(3) $\dim \mathcal{S}_a - \dim \mathcal{B}_a \le R$.

Proof. By the assumption, there is a morphism $\mathcal{Y} \to \mathcal{T}$ between algebraic schemes such that every $\mathbb{Q}$-Fano n-folds with Picard number one is birational to one of the geometric fibers of $\mathcal{Y} \to \mathcal{T}$. As $\mathcal{T}$ has only finitely many components, there is at least one component, say $\mathcal{T}_a$, such that a sufficiently general member of $\mathcal{X}_a/\mathcal{S}_a$ is birational to one of the geometric fibers of $\mathcal{Y}_a := \mathcal{Y} \times_{\mathcal{T}} \mathcal{T}_a \to \mathcal{T}_a$. Without loss of generality, we may assume that the morphism $\mathcal{Y}_a \to \mathcal{T}_a$ is flat and projective. Let

$$\pi \colon \Gamma := \Gamma(\mathcal{X}_a/\mathcal{S}_a, \mathcal{Y}_a/\mathcal{T}_a) \to \mathcal{H} := \mathrm{Bir}(\mathcal{X}_a/\mathcal{S}_a, \mathcal{Y}_a/\mathcal{T}_a)$$

be the universal family of birational correspondences between members of $\mathcal{X}_a/\mathcal{S}_a$ and $\mathcal{Y}_a/\mathcal{T}_a$. By the definition, $\mathcal{H}$ is an open subscheme of $\mathrm{Hilb}(\mathcal{X}_a \times \mathcal{Y}_a/\mathcal{S}_a \times \mathcal{T}_a)$ and Γ is a closed subscheme of $(\mathcal{X}_a \times \mathcal{Y}_a) \times_{(\mathcal{S}_a \times \mathcal{T}_a)} \mathcal{H}$. Let $q_1 \colon \mathcal{H} \to \mathcal{X}_a$ (resp. $q_2 \colon \mathcal{H} \to \mathcal{Y}_a$) be the composite of $p \colon \mathcal{H} \to \mathcal{S}_a \times \mathcal{T}_a$ and the natural projection $\mathcal{S}_a \times \mathcal{T}_a \to \mathcal{S}_a$ (resp. $\mathcal{S}_a \times \mathcal{T}_a \to \mathcal{T}_a$). By the assumption and our choice of $\mathcal{T}_a$, we see that q_1 is dominant. As $\mathcal{H}$ has at most countably many irreducible components, we may assume, after replacing $\mathcal{H}$ by its suitable irreducible component, that $\mathcal{H}$ is a variety and $q_1 \colon \mathcal{H} \to \mathcal{S}_a$ is still dominant. Let t be a general point of $\mathrm{Im}(q_2) \subset \mathcal{T}_a$ and $\mathcal{H}_t$ the fiber of q_2 over t. By the construction, every member parametrized by $\mathcal{B}_a := q_1(\mathcal{H}_t)$ is birational to $\mathcal{Y}_t$. As q_1 is dominant, we see that $\mathcal{B}_a$ passes through a general point $s_a \in \mathcal{S}_a$. We have $\dim \mathcal{T} \ge \dim \mathcal{H} - \dim \mathcal{H}_t$, $\dim \mathcal{H} = \dim \mathcal{S}_a + \dim(q_1)$ and $\dim \mathcal{H}_t = \dim \mathcal{B}_a + \dim(q_1|_{\mathcal{H}_t})$, where $\dim(q)$ is the dimension of the generic fiber for a morphism q of finite type. Therefore, we see that

$$\dim \mathcal{S}_a - \dim \mathcal{B}_a \le \dim \mathcal{S}_a - \dim \mathcal{B}_a + (\dim(q_1) - \dim((q_1)|_{\mathcal{H}_t}) \le \dim \mathcal{T}.$$

We have only to put $R := \dim \mathcal{T}$. This completes the proof. □

Let $f = f(x_0, \dots, x_n) \in H_{al}(\mathbb{C})$ be a very general element and put $X = X_f$. There is a discrete valuation ring R such that it is a localization of a finitely generated $\mathbb{Z}$-algebra, its residue field has characteristic two and X descends to a scheme $\mathfrak{X}$ over R. Let us briefly explain the construction of R. We refer the reader to [9, Section 4.4] for a detail. We can write

$$f = \sum\nolimits_{\Lambda} c_\Lambda x^\Lambda,$$

where $\Lambda = (\lambda_0, \dots, \lambda_n)$, $c_\Lambda \in \mathbb{C}$ and $x^\Lambda = x_0^{\lambda_0} \cdots x_n^{\lambda_n}$. We may assume that the c_Λ's are algebraically independent over $\mathbb{Q}$ because f is very general. Let $S := \mathbb{Z}[\{ c_\Lambda \}]$ be the subring of $\mathbb{C}$. Then the localization R of S at its prime ideal

(2) is a discrete valuation ring and the residue field of R has characteristic two. It follows from the construction that X descends to the scheme

$$\mathfrak{X} = \mathfrak{X}_f := (y^2 x_0 - f = 0) \subset \mathbb{P}_R(\overbrace{1,\dots,1}^{m+1}, \overbrace{a,\dots,a}^{n-m}, b)$$

over $\operatorname{Spec} R$. Let X' be the geometric special fiber of $\mathfrak{X} \to \operatorname{Spec} R$ so that it is a member of $\mathcal{X}'_a/\mathcal{S}'_a$. By replacing R if necessary, we assume that the isolated hypersurface singularities (cf. Lemma 2.6) on X' are defined on $\mathfrak{X}$.

Lemma 3.5. *Notation and assumption as above. There is a resolution $\rho\colon \widetilde{\mathfrak{X}} \to \mathfrak{X}$ of singularities of $\mathfrak{X}$ such that, for every exceptional divisor E of ρ, its special fiber E_{sp} has dimension less than n or E_{sp} is ruled.*

Proof. The scheme $\mathfrak{X}$ is a closed subscheme of $P_R := \mathbb{P}_R(1,\dots,1,a,\dots,a,b)$. Let $t \in R$ be a uniformizing parameter of R. By choosing suitable local coordinates of P_R, the singularity of $\mathfrak{X}$ on $\mathrm{D}_+(x_0)$ is isomorphic to the point, which corresponds to the maximal ideal $(t, \xi_1, \dots, \xi_n, \nu)$, of the hypersurface determined by the equation

$$\nu^2 = t\alpha + th_1 + + \xi_1\xi_2 + \xi_3\xi_4 + \cdots + \xi_{n-1}\xi_n + th_2 + h_{\geq 3}$$

if n is even, or

$$\nu^2 = t\alpha' + th'_1 + \beta'\xi_1^2 + \xi_2\xi_3 + \xi_4\xi_5 + \cdots + \xi_{n-1}\xi_n + th'_2 + \gamma'\xi_1^3 + h'_{\geq 3}$$

if n is odd, where $\alpha, \alpha'\beta' \in R$, $\gamma' \in R^\times$, $h_i, h'_i \in R[\xi_1, \dots, \xi_n]$ are polynomials of degree i and $h_{\geq 3}, h'_{\geq 3} \in R[\xi_1, \dots, \xi_n]$ are polynomials which consists of monomials of degree ≥ 3. This singularity can be resolved by blowing up the point. Let E be the exceptional divisor of the blow up. It is straightforward to check that $E = E_{\mathrm{sp}}$ is the cone over a quadric. In particular, it is ruled.

There is a desingularization of the toroidal embedding $U_{\mathrm{qs}} \subset X_{\mathrm{qs}}$ (cf. Lemma 2.6). Such a morphism is defined over R and we obtain a birational morphism $\rho_1\colon \mathfrak{X}_1 \to \mathfrak{X}$. Let E be an exceptional divisor of ρ_1. Then, we have $\dim E_{\mathrm{sp}} = \dim E - 1 \leq n - 1$. Let $\rho\colon \widetilde{\mathfrak{X}} \to \mathfrak{X}$ be the composite of ρ_1 and blowing ups at each isolated hypersurface singular points. This completes the proof. $\square$

Lemma 3.6. *Let l, m and n be integers which satisfy* Condition 2.3*, and let a be an odd integer with $a > m+1$. Let $f, g \in H_{al}(\mathbb{C})$ be very general elements. If X_f is birational to X_g over $\mathbb{C}$ then X'_f is birational to X'_g, where X'_f and X'_g are reduction mod 2 models of X_f and X_g respectively.*

Proof. Let $\psi\colon X_f \dashrightarrow X_g$ be a birational map. There is a discrete valuation ring R with the following properties.

- R is a localization of a finitely generated $\mathbb{Z}$-algebra and its residue field has characteristic two.
- X_f and X_g descend to schemes $\mathfrak{X}_f$ and $\mathfrak{X}_g$ over R respectively.
- The birational map $\psi\colon X_f \dashrightarrow X_g$ descends to a birational map $\Psi\colon \mathfrak{X}_f \dashrightarrow \mathfrak{X}_g$.

The geometric special fibers of $\mathfrak{X}_f$ and $\mathfrak{X}_g$ are isomorphic to X'_f and X'_g respectively. After replacing R, we may assume that the isolated singular points of X'_f and X'_g on $\mathrm{D}_+(x_0)$ are defined on $\mathfrak{X}_f$ and $\mathfrak{X}_g$ respectively.

Let $\rho_f\colon \widetilde{\mathfrak{X}}_f \to \mathfrak{X}_f$ and $\rho_g\colon \widetilde{\mathfrak{X}}_g \to \mathfrak{X}_g$ be the resolution of singularities of $\mathfrak{X}_f$ and $\mathfrak{X}_g$ respectively, which are obtained by Lemma 3.5. Let $\tilde{\Psi}\colon \widetilde{\mathfrak{X}}_f \dashrightarrow \widetilde{\mathfrak{X}}_g$ be the birational which is induced by Ψ. Let $\widetilde{X'}_f$ and $\widetilde{X'}_g$ be the strict transform of X'_f and X'_g in $\widetilde{\mathfrak{X}}_f$ and $\widetilde{\mathfrak{X}}_g$ respectively. The birational map $\tilde{\Psi}$ does not contracts $\widetilde{X'}_f$ because it is not ruled by Theorem 2.4. Therefore, by Lemma 3.5, $\tilde{\Psi}$ induces a birational map between $\widetilde{X'}_f$ and $\widetilde{X'}_g$. This shows that X'_f and X'_g are birational. □

Proposition 3.7. *Suppose that the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally bounded. Then, there exists a constant R' such that, for every odd integer a with $a > m+1$ and a general point $s'_a \in \mathcal{S}'_a$, there is a closed subvariety $\mathcal{B}'_a$ of $\mathcal{S}'_a$ with the following properties.*

(1) *$\mathcal{B}'_a$ parametrizes a birationally trivial family.*
(2) *$\mathcal{B}'_a$ passes through s'_a.*
(3) $\dim \mathcal{S}'_a - \dim \mathcal{B}'_a \leq R'$.

Proof. Put $\mathcal{X} = \mathcal{X}_a$ and $\mathcal{S} = \mathcal{S}_a$. By Lemma 3.4, there is a closed subvariety $\mathcal{B} \subset \mathcal{S}$ which parametrizes a birationally trivial family and $\dim \mathcal{S} - \dim \mathcal{B} \leq R$. Let

$$\pi\colon \Gamma_{\mathcal{B}} := \Gamma(\mathcal{X}_{\mathcal{B}}/\mathcal{B}, \mathcal{X}_{\mathcal{B}}/\mathcal{B}) \to \mathcal{H}_{\mathcal{B}} := \mathrm{Bir}(\mathcal{X}_{\mathcal{B}}/\mathcal{B}, \mathcal{X}_{\mathcal{B}}/\mathcal{B})$$

be the universal family of birational correspondences between two copies of the family $\mathcal{X}_{\mathcal{B}}/\mathcal{B}$. We see that $p\colon \mathcal{H}_{\mathcal{B}} \to \mathcal{B}\times\mathcal{B}$ is surjective since $\mathcal{X}_{\mathcal{B}}/\mathcal{B}$ is a birationally trivial family. Without loss of generality, we may assume that $\mathcal{H}_{\mathcal{B}}$ is irreducible.

Let $\mathcal{B}' = \mathcal{B}'_a$ be the reduction mod 2 model of $\mathcal{B}$. Let $\Gamma_{f,g} \subset X_f \times X_g$ be a birational correspondence between two general members X_f and X_g of $\mathcal{X}_{\mathcal{B}}/\mathcal{B}$, which corresponds to a general point of $\mathcal{H}_{\mathcal{B}}$. Let $\varphi\colon X_f \dashrightarrow X_g$ be the birational map induced by the birational correspondence above. By Lemma 3.6, the birational map φ induces a birational map between reduction 2 models of X_f and X_g. This shows that, after shrinking $\mathcal{S}'$ and then $\mathcal{B}'$ if necessary, we see that $\mathcal{B}'$ parametrizes a birationally trivial family. Finally, we see that

$$\dim \mathcal{S}' - \dim \mathcal{B}' \leq \dim \mathcal{S} - \dim \mathcal{B} \leq R,$$

since we have $\dim \mathcal{S}' = \dim \mathcal{S}$ and $\dim \mathcal{B}' \geq \dim \mathcal{B}$. Put $R' = R$. This completes the proof. □

4. Bounding birationally trivial families

In this section, we shall count the dimension of birationally trivial subfamilies of the family $\mathcal{X}'_a/\mathcal{S}'_a$ and prove Theorem 1.2. Throughout the present section,

- we work over an algebraically closed field $\Bbbk$ of characteristic two,
- we fix positive integers l, m, and n which satisfy the inequality $l + 2m - 2n + 2 \leq 0$ in addition to Condition 2.3, and
- $f = f(x_0, \dots, x_n)$ and $g = g(x_0, \dots, x_n)$ are both general elements of $H_{al}(\Bbbk)$.

Definition 4.1. We denote by G the subgroup

$$G := \{\sigma \in \mathrm{Aut}(Q) \mid \sigma^* x_0 = \alpha x_0 \text{ for some } \alpha \in \Bbbk^\times\}$$

of the group of automorphisms of Q.

Lemma 4.2. *Suppose that there is a birational map $\varphi\colon X_f \dashrightarrow X_g$. Then, there is an isomorphism $\sigma\colon Q \to Q$ such that $\sigma \in G$ and the diagram*

$$\begin{tikzcd} X_f \arrow[r, dotted, "\varphi"] \arrow[d, dotted, "\pi_f"'] & X_g \arrow[d, dotted, "\pi_g"] \\ Q \arrow[r, "\sigma"'] & Q \end{tikzcd}$$

commutes.

Proof. We fix a resolution $Y_f \to X_f$ (resp. $Y_g \to X_g$) of singularities of X_f (resp. X_g). Let $\mathcal{M}_f$ (resp. $\mathcal{M}_g$) be the reflexive sheaf of rank one on X_f (resp. X_g) defined in Definition 2.9 and $\mathcal{L}_f$ (resp. $\mathcal{L}_g$) be the invertible sheave on Y_f (resp. Y_g) defined in Definition 2.11. Let $\psi\colon Y_f \dashrightarrow Y_g$ be the birational map induced by φ. Let Z_f (resp. Z_g) be the image of the rational map $\Phi_{|\mathcal{L}_f|}$ (resp. $\Phi_{|\mathcal{L}_g|}$). By Lemma 2.15 and the fact that $H^0(Y_f, \Omega^{n-1}_{Y_f}) \cong H^0(Y_g, \Omega^{n-1}_{Y_g})$, there is a natural isomorphism $\gamma\colon Z_f \to Z_g$ such that the diagram

$$\begin{tikzcd} Y_f \arrow[r, dotted, "\psi"] \arrow[d, dotted, "\Phi_{|\mathcal{L}_f|}"'] & Y_g \arrow[d, dotted, "\Phi_{|\mathcal{L}_g|}"] \\ Z_f \arrow[r, "\gamma"'] & Z_g \end{tikzcd}$$

commutes. It follows from Lemma 2.13 that Z_f and Z_g are blow ups of Q along the subvariety $(x_0 = \cdots = x_m = 0)$. Hence, the isomorphism $\gamma\colon Z_f \to Z_g$ descends to an isomorphism $\sigma\colon Q \to Q$. Therefore, we obtain a commutative diagram

$$\begin{tikzcd} X_f \arrow[rrr, dotted, "\varphi"] \arrow[dr, dotted, "\Phi_{|\mathcal{M}_f|}"] \arrow[dd, dotted, "\pi_f"'] & & & X_g \arrow[dl, dotted, "\Phi_{|\mathcal{M}_g|}"'] \arrow[dd, dotted, "\pi_g"] \\ & Z_f \arrow[r, "\gamma"] & Z_g & \\ Q \arrow[rrr, "\sigma"'] \arrow[ur, dotted, "\Phi_{|\mathcal{O}_Q(A)|}"] & & & Q. \arrow[ul, dotted, "\Phi_{|\mathcal{O}_Q(A)|}"'] \end{tikzcd}$$

Let D_f (resp. D_g) be the hypersurface of X_f (resp. X_g) cut out by x_0 and H be the zero locus of x_0 in Q. We see that D_f (resp. D_g) is the only divisor which is contracted by π_f (resp. π_g). Suppose that σ is not contained in G. Then, the divisors H and σ^*H on Q are distinct. Pick a divisor D'_f on X_f which dominates σ^*H. Then $\varphi_* D'_f$ must be a divisor on X_g dominating H. This is a contradiction because there is no divisor on X_g dominating H. Therefore, we have $\sigma \in G$ and this completes the proof. □

Lemma 4.3. *Suppose that there is a birational map $\varphi\colon X_f \dashrightarrow X_g$ and let $\sigma \in G$ be the automorphism of Q obtained by Lemma 4.2. Then there is an automorphism φ_P of P with the following properties.*

(1) *$\pi_P \circ \varphi_P = \sigma \circ \pi_P$, where π_P is the natural projection $\pi_P\colon P \dashrightarrow Q$.*

(2) *$(\varphi_P)^* g = \sigma^* g = \beta(f + x_0 h^2)$ for some $\beta \in \Bbbk^\times$ and $h \in H_b(\Bbbk)$.*
(3) *The restriction of φ_P on X_f defines an isomorphism $(\varphi_P)|_{X_f} \colon X_f \to X_g$ between X_f and X_g, and it coincides with φ.*

In particular, φ is an isomorphism.

Proof. We consider the weighted hypersurface

$$\widetilde{Q}_f := (\tilde{y}^2 - x_0 f = 0) \subset \widetilde{P} := \mathbb{P}(1^{m+1}, a^{n-m}, b+1).$$

Let $x_0, \dots, x_n$ and $\tilde{y}$ be the homogeneous coordinates of $\widetilde{P}$ with the gradings $\deg x_i = 1$ for $0 \le i \le m$, $\deg x_i = a$ for $m+1 \le i \le n$ and $\deg \tilde{y} = b+1$. Let $\Bbbk[x_0, \dots, x_n, \tilde{y}] \to \Bbbk[x_0, \dots, x_n, y]$ be the homomorphism of graded rings defined by $x_i \mapsto x_i$ and $\tilde{y} \mapsto y x_0$. This defines a birational map $P \dashrightarrow \widetilde{P}$ and then a birational map $X_f \dashrightarrow \widetilde{Q}_f$. We see that $\widetilde{Q}_f$ is the normalization of Q in the function field of X_f. By Lemma 4.2, there is an isomorphism $\tilde{\sigma} \colon \widetilde{Q}_f \to \widetilde{Q}_g$. We can write $\tilde{\sigma}^* \tilde{y} = \gamma \tilde{y} + h'$ for some $\gamma \in \Bbbk$ and $h' \in \Bbbk[x_0, \dots, x_n]_{b+1}$. We have

$$\tilde{\sigma}^*(\tilde{y}^2) = \tilde{\sigma}^*(x_0 g) = \sigma^*(x_0 g) = \alpha x_0 \sigma^* g,$$

where α is an element of $\Bbbk^\times$ such that $\sigma^* x_0 = \alpha x_0$. On the other hand, we have

$$\tilde{\sigma}^*(\tilde{y}^2) = (\gamma \tilde{y} + h')^2 = \gamma^2 \tilde{y}^2 + h'^2 = \gamma^2 x_0 f + h'^2.$$

Thus, we see that $h'^2 = \alpha x_0 \sigma^* g - \gamma^2 x_0 f$ and we can write $h' = x_0 h$ for suitable $h \in H_b(\Bbbk)$. Hence we have $\alpha \sigma^* g = \gamma^2 f + x_0 h^2$ and this implies that $\gamma \neq 0$ since g is general and σ is an automorphism. This shows that the isomorphism $\tilde{\sigma}$ lifts to the isomorphism $\tilde{\sigma}_{\widetilde{P}} \colon \widetilde{P} \to \widetilde{P}$ determined by $(\tilde{\sigma}_{\widetilde{P}})^* x_i = \sigma^* x_i$ and $(\tilde{\sigma}_{\widetilde{P}})^* \tilde{y} = \gamma \tilde{y} + x_0 h$.

Now let φ_P be the automorphism of P determined by $(\varphi_P)^* x_i = \sigma^* x_i$ and $(\varphi_P)^* y = (\gamma y + h)/\alpha$. This defines an isomorphism $(\varphi_P)|_{X_f} \colon X_f \to X_g$ and, by the construction, it coincides with the birational map $\varphi \colon X_f \dashrightarrow X_g$. $\square$

Lemma 4.4. *We have* $\dim(\mathrm{Aut}(Q)) = (m+1)^2 + (n-m) h^0(\mathcal{O}_Q(a)) - 1$.

Proof. Put $e := h^0(\mathcal{O}_Q(a))$. Let $A \in \mathrm{GL}(m+1) = \mathrm{GL}(m+1, \Bbbk)$, $B \in \mathrm{GL}(n-m) = \mathrm{GL}(n-m, \Bbbk)$ and $C \in \mathrm{M} = \mathrm{M}(n-m, e-n+m, \Bbbk)$ be matrices. The matrix A defines an automorphism ρ_A of $\Bbbk[x_0, \dots, x_n]_1 = \Bbbk \cdot x_0 + \cdots + \Bbbk \cdot x_m$ and the matrix

$$\begin{pmatrix} \mathrm{Sym}^a(A) & O \\ C & B \end{pmatrix} \in \mathrm{GL}(e)$$

defines an automorphism $\tau_{A,B,C}$ of $\Bbbk[x_0, \dots, x_n]_a$ such that $(\tau_{A,B,C})|_{\Bbbk[x_0, \dots, x_m]_a}$ coincides with $\mathrm{Sym}^a(\rho_A)$, where $\mathrm{Sym}^a(A)$ is the matrix which represents the automorphism $\mathrm{Sym}^a(\rho_A)$ and O is the zero matrix. The pair $(\rho_A, \tau_{A,B,C})$ of automorphisms defines an automorphism of the graded ring $\Bbbk[x_0, \dots, x_n]$ in a natural way and thus an automorphism $\sigma_{A,B,C}$ of Q. Notice that we have $\sigma_{A,B,C} = \sigma_{A',B',C'}$ if and only if there is some $\alpha \in \Bbbk^\times$ such that $(A, B, C) = (\alpha A', \alpha^a B', \alpha^a C')$. This shows that there is a morphism

$$\mathrm{GL}(m+1) \times \mathrm{GL}(n-m) \times \mathrm{M} \to \mathrm{Aut}(Q),$$

which sends (A, B, C) to $\sigma_{A,B,C}$. It can be checked that the morphism above is surjective. Let $\{u_{ij}\}, \{v_{ij}\}$ and $\{w_{ij}\}$ be the system of affine coordinates of

$\mathrm{GL}(m+1), \mathrm{GL}(n-m)$ and M respectively. We consider the $\mathbb{G}_{\mathtt{m}} = \operatorname{Spec} \Bbbk[t, t^{-1}]$ action on $\mathrm{GL}(m+1) \times \mathrm{GL}(n-m) \times \mathrm{M}$ defined by

$$u_{ij} \mapsto u_{ij} \otimes t, v_{ij} \mapsto v_{ij} \otimes t^a, w_{ij} \mapsto w_{ij} \otimes t^a.$$

Then we see that there is a morphism

$$(\mathrm{GL}(m+1) \times \mathrm{GL}(n-m) \times \mathrm{M}) / \mathbb{G}_{\mathtt{m}} \to \mathrm{Aut}(Q)$$

and it is an isomorphism. Thus, we see that

$$\begin{aligned} \dim \mathrm{Aut}(Q) &= (m+1)^2 + (n-m)^2 + (n-m)(e-n+m) - 1 \\ &= (m+1)^2 + (n-m)e - 1. \end{aligned}$$

This completes the proof. □

Definition 4.5. Let V be the $\Bbbk$-vector space $H_{al}(\Bbbk)$. We denote by V' the $\Bbbk$-vector subspace

$$V' = \{ x_0 h^2 \mid h \in H_b(\Bbbk) \}$$

of V. For an element $f \in V$, we denote by V_f the subset

$$V_f := \left\{ g \in V \mid g = \beta(\sigma^* f + x_0 h^2) \text{ for some } \beta \in \Bbbk^{\times}, \sigma \in G \text{ and } h \in H_b(\Bbbk) \right\}$$

of V.

Notice that we have $f \in V_f$.

Proposition 4.6. *Let f and g be general elements of $H_{al}(\Bbbk)$. Then, the following statements are equivalent.*

(1) *X_f is isomorphic to X_g.*
(2) *X_f is birational to X_g.*
(3) *$V_f = V_g$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious and the implication (2) $\Rightarrow$ (3) is proved in Lemma 4.3. We shall prove that (3) implies (1). Suppose that $V_f = V_g$. Then $g \in V_g$ implies $g \in V_f$, so that there are $\alpha, \beta \in \Bbbk^{\times}$, $\sigma \in G$ and $h \in H_b(\Bbbk)$ such that $\sigma^* x_0 = \alpha x_0$ and $g = \beta(\sigma^* f + x_0 h^2)$. Let φ_P be the automorphism determined by $(\varphi_P)^* x_i = \sigma^* x_i$ and $(\varphi_P)^* y = (1/\sqrt{\alpha\beta}) y + (1/\sqrt{\alpha}) h$. This defines an isomorphism $\varphi_P|_{X_g} \colon X_g \to X_f$ and the implication (3) $\Rightarrow$ (1) is proved. □

Proposition 4.7. *For any odd integer $a > m+1$ and a general point $s'_a \in \mathcal{S}'_a$, there is a closed subscheme $\mathcal{C}'_a$ of $\mathcal{S}'_a$ with the following properties.*

(1) *$\mathcal{C}'_a$ parametrizes the members which are birational to the member $\mathcal{X}'_{a,s'_a}$ corresponds to s'_a.*
(2) $\dim \mathcal{S}'_a - \dim \mathcal{C}'_a \to +\infty \quad (a \to +\infty)$.

Proof. We see that G naturally acts on $\mathbb{P}_{\mathrm{sub}}(V)$, that is, there is a morphism

$$G \times \mathbb{P}_{\mathrm{sub}}(V) \to \mathbb{P}_{\mathrm{sub}}(V),$$

which sends $(\sigma, [f])$ to $[\sigma^* f]$, where $[f]$ is the point of $\mathbb{P}_{\mathrm{sub}}(V)$ corresponds to $f \in V$. Now let f be an element of V such that $s'_a = [f] \in \mathbb{P}_{\mathrm{sub}}(V)$ and let $\overline{G \cdot [f]}$ be the closure of the image of $G \times \{[f]\}$. Let $\mathbb{P}_{\mathrm{sub}}(V) \dashrightarrow \mathbb{P}_{\mathrm{sub}}(V/V')$ be the projection from the linear subspace $\mathbb{P}_{\mathrm{sub}}(V') \subset \mathbb{P}_{\mathrm{sub}}(V)$ and let $\mathcal{C}'_a$ be the cone over the image of $\overline{G\cdot[f]}$ under the projection. It follows that $\mathcal{C}'_a$ contains every point of V_f. By Proposition 4.6, we have (1).

By Lemma 4.4, we have

$$\begin{aligned}\dim \mathcal{C}'_a &\le \dim \overline{G\cdot[f]} + \dim V' \le \dim G + \dim V' \\ &< (n-m)h^0(\mathcal{O}_Q(a)) + h^0(\mathcal{O}_Q(b)) + (m+1)^2.\end{aligned}$$

We can calculate $h^0(\mathcal{O}_Q(al)), h^0(\mathcal{O}_Q(a))$ and $h^0(\mathcal{O}_Q(b))$ explicitly and we have the following estimates.

- $h^0(\mathcal{O}_Q(al)) = a^m \sum_{k=0}^{l-1} \frac{(n-m+k-1)!}{(n-m-1)!k!}\frac{(l-k)^m}{m!} + O(a^{m-1})$.
- $h^0(\mathcal{O}_Q(a)) = \frac{1}{m!}a^m + O(a^{m-1})$.
- $h^0(\mathcal{O}_Q(b)) = a^m \sum_{k=0}^{(l-1)/2} \frac{(n-m+k-1)!}{(n-m-1)!k!}\frac{(l-2k)^m}{m!2^m} + O(a^{m-1})$.

Thus, we have

$$\begin{aligned}\dim \mathcal{S}'_a - \dim \mathcal{C}'_a &\ge h^0(\mathcal{O}_Q(al)) - (n-m)h^0(\mathcal{O}_Q(a)) - h^0(\mathcal{O}_Q(b)) + O(a^{m-1}) \\ &\ge \left(\frac{(n-m+l-2)!}{(n-m-1)!(l-1)!}\frac{1}{m!} - \frac{n-m}{m!}\right)a^m + O(a^{m-1}).\end{aligned}$$

Condition 2.3 in particular implies that $n-m \ge 2$, which shows that the coefficient of a^m in the inequality above is positive. This proves (2). $\square$

of Theorem 1.2. If $n \ge 6$, then we can find integers l, m and n which satisfy Condition 2.3 and the inequality $l + 2m - 2n + 2 \le 0$. We fix such l, m and n. We assume that the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally bounded. Then, by Proposition 3.7, there is a constant R' and a closed subvariety $\mathcal{B}'_a$ of $\mathcal{S}'_a$ which passes through a general point $s'_a \in \mathcal{S}'_a$ and parametrizes a birationally trivial family such that $\dim \mathcal{S}'_a - \dim \mathcal{B}'_a \le R'$. Let $\mathcal{C}'_a$ be a subscheme of $\mathcal{S}'_a$ obtained by Proposition 4.7. We may assume that $\mathcal{C}'_a$ passes through s'_a so that it contains $\mathcal{B}'_a$ by the property (1) of Proposition 4.7. Therefore, we have

$$\dim \mathcal{S}'_a - \dim \mathcal{C}'_a \le \dim \mathcal{S}'_a - \dim \mathcal{B}'_a \le R'.$$

This contradicts to the property (2) of Proposition 4.7. $\square$

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